Memories and notes presented by fellows

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Mathematics. – On conformal representations, and on a physical interpretation of the Levi-Civita parallelism.

1. The scope of this Note is to provide a physical interpretation of the Levi-Civita parallelism within the field of classical Physics, and even if [*such interpretation will be*] limited to the particular case in which the [*mathematical*] variety is a conformally representable one, V, on a euclidean [*space*] $\mathbb{R}_3$ – that is, we will follow Schouten calling it a $\mathbb{C}_3$ [2] – and even if [*such interpretation will be limited to the case in which*] the transported vector is perpendicular to the direction of transport, it still seems to me not lacking of interest.

2. I will first make some introductory considerations with regard to conformal transformations. It is known [3] that if $\bar{V}_n$ is conformally represented on V_n through the transformation $\bar{a}_{ik} = n^2 a_{ik}$ ($\bar{a}_{ik}$, a_{ik} are fundamental tensors of the two varieties), setting $\mathbf{p} = \operatorname{grad}\log n$, using $\bar{\delta}$, δ as symbols of covariant differentiation in $\bar{V}_n$ and V_n, [*respectively,*] and using λ for the unitary vector in V_n that gives the direction of the derivative, we have

$$\frac{\bar{\delta}\xi^i}{\delta s_\lambda} = \frac{\delta\xi^i}{\delta s_\lambda} - \xi\times\lambda\cdot p^i + \mathbf{p}\times\lambda\cdot\xi^i + \mathbf{p}\times\xi\cdot\lambda^i,$$
$$\frac{\bar{\delta}\xi_i}{\delta s_\lambda} = \frac{\delta\xi_i}{\delta s_\lambda} - \xi\times\lambda\cdot p_i - \mathbf{p}\times\lambda\cdot\xi_i + \mathbf{p}\times\xi\cdot\lambda_i, \tag{1}$$

where the scalar products are meant as calculated in the metric of V_n. If ξ is a unitary vector in V_n, assuming $\bar{\xi}^i = \frac{1}{n}\xi^i$, $\bar{\xi}_i = n\xi_i$, $\bar{\xi}$ is unitary in $\bar{V}_n$ and we have

$$n\frac{\bar{\delta}\bar{\xi}^i}{\delta s_\lambda} = \frac{\delta\xi^i}{\delta s_\lambda} - \left(\xi\times\lambda\cdot p^i - \mathbf{p}\times\xi\cdot\lambda^i\right),$$
$$\frac{1}{n}\frac{\bar{\delta}\bar{\xi}_i}{\delta s_\lambda} = \frac{\delta\xi_i}{\delta s_\lambda} - \left(\xi\times\lambda\cdot p_i - \mathbf{p}\times\xi\cdot\lambda_i\right). \tag{2}$$

The condition for which ξ gets transported in V_n through a parallelism in the direction (λ) is thus that the corresponding vector ξ of V_n satisfies the vectorial equation

$$\frac{\delta\xi}{\delta s_\lambda} = \xi\times\lambda\cdot\mathbf{p} - \mathbf{p}\times\xi\cdot\lambda, \tag{3}$$

which in the particular case $n = 3$ takes the form

$$\frac{\delta\xi}{\delta s_\lambda} = (\lambda\wedge\mathbf{p})\wedge\xi. \tag{3*}$$

For $\xi \equiv \lambda$ we obtain in particular from Eq. (3) the differential equation of the lines of V_n that correspond to the geodesics of $\bar{V}_n$. If the two varieties V_n, $\bar{V}_n$ in conformal representation are obtained by assigning two different metrics to the same abstract variety, Eq. (3) represents, in V_n, the Levi-Civita parallelism with respect to the metric of $\bar{V}_n$, and for $\xi \equiv \lambda$, it defines in V_n the geodesics in $\bar{V}_n$. This indeed shows us that it is characteristic of the geodesics of $\bar{V}_n$, considered in V_n, to have at every point the G_2 (flat affine space) osculating normal to the level ipersurface of level $n =$cost. passing by that point; that is, containing the vector $\mathbf{p}$. Moreover, from Eq. (3) we have a simple construction, in V_n, of the series of parallel directions in V_n exiting orthogonally from a geodesic Γ of V_n. Precisely: if Σ_ξ is the *stripe* that is obtained transporting ξ through Levi-Civita parallelism in V_n along Γ, then the direction (ξ^*) parallel to (ξ) in $\bar{V}_n$, through infinitesimal transport from P to $P^* = P + \lambda ds_\lambda$ along Γ, is the intersection of Σ_ξ with the G_{n-1} normal to Γ in P^*.

In particular, if $n = 3$ and V_n is an euclidean space $\mathbb{R}_3$ – from which it follows that V_n is a $\mathbb{C}_3$ – then we have (always for $\xi\times\lambda = 0$) that the parallel (in $\mathbb{C}_3$) to ξ exiting from P^* is the direction that is orthogonally projected on the plane normal in P to Γ, in the direction ξ, that is the *intersection of the plane $\xi\lambda$ with the plane normal in P^* to Γ.*

3. I will here shortly recall some known results of Optics. Let us consider a linearly polarized luminous ray, along which, that is, the vibrations happen in only one direction (ξ), normal to the ray; and let us suppose that it transverses the surface of separation [*interface*] of two isotropic media, with refraction indexes n_1, n_2. In refraction, the direction of the luminous vector gets rotated; precisely, if ξ' is the luminous vector relative to the refracted ray, λ, λ' the unitary vectors oriented like the incident and refracted rays, (ζ) the normal to the surface of separation of the two media in the incident point, setting $\nu = \operatorname{vers}(\lambda\wedge\zeta)$, $\alpha = \widehat{\nu\xi}$, $\alpha' = \widehat{\nu\xi'}$, and $i = \widehat{\lambda\zeta}$, $r = \widehat{\lambda\zeta'}$ (the angles of incidence and refraction), we have

$$\cot\alpha' = \cot\alpha\cos(i-r)[4]. \tag{4}$$

Now Eq. (4) can be interpreted in the following way:

(ξ') *is the intersection of the plane $\lambda\xi$ with the plane normal in P to the refracted ray.*

The effective expressions of ξ and ξ' (unitary luminous vectors) for λ, ζ, α, i, and r are easily obtained: once the appropriate and obvious conventions on the meaning of the vectors are made, we have

$$\xi = \frac{\sin\alpha(\zeta - \cos i\lambda) + \cos\alpha\lambda\wedge\zeta}{\sin i}$$
$$\xi' = \frac{\cos(i-r)}{\sqrt{1-\cos^2\alpha\sin^2(i-r)}}\cdot\frac{\sin\alpha\left(\zeta - \frac{\cos r}{\cos(i-r)}\lambda\right) + \cos\alpha\lambda\wedge\zeta}{\sin i} \tag{5}$$

4. All that set forth, I will come to the aforementioned interpretation. It is known [5] that the geometrical optics of a medium N of refraction index $n(x, y, z)$, continuously variable from point to point, can be expressed with the single variational formula

$$\delta \int nds = 0, \tag{6}$$

(where $ds^2 = dx^2 + dy^2 + dz^2$), illustrating Fermat's principle of least time [6]. The trajectories of the luminous rays in such medium are the extremal [*paths*] of Eq. (6). It is evident how one could consider, in the $\mathbb{R}_3$ space, in which these phenomena occur, instead of the euclidean metric, the metric that has for linear element $dt = nds$ (that is, a metric [*that is*] *euclideanly connected* and *without torsion*, according to Cartan, or *Riemannian* metric according to Schouten): the space then becomes a $\mathbb{C}_3$, of which the luminous rays are the geodesics. That is equivalent to choosing as measure of the length of the portion of luminous ray comprised between two points, the corresponding optical path – or equivalently, the time that takes light to cover it. Now:

If in the considered medium N a linerarly polarized ray propagates, along such ray a luminous vector is transported through the Levi-Civita parallelism in $\mathbb{C}_3$.

As an effect [*of this*]: we notice that [Levi-Civita, Duhem, *loc. cit*] that the medium N can be considered as formed by many infinitesimally thin layers with constant refraction index (for each of them), which are inserted between two subsequent level surfaces of level n =const. Now: if P, P^* are the points in which a generic ray Γ (to which λ gives in the generic point, the orientation) crosses two subsequent level surfaces, it becomes obvious, from what we illustrated, that the construction with which from the luminous vector ξ, relative to P, one obtains the vector ξ^*, relative to P^*, it is exactly the same that gives the vector parallel to P^* in the sense of Levi-Civita in $\mathbb{C}_3$, to the vector ξ exiting from P through transport along Γ.

After all, in the hypothesis in which $n_2 - n_1$ is [*an*] infinitesimal [*quantity*], and precisely in the case in which $n_1 = n$, $n_2 = n + \Delta n$ $(= n(1 + M \cos i\Delta s)$, where $M = \text{mod}\mathbf{p} = \sqrt{\Delta_I(\log n)}$ and $P^* = P + \lambda\Delta s)$, from which it follows that the principal part of $i - r$ is $M \sin i\Delta s$, we easily obtain from Eq. (5) the differential formula

$$\frac{d\xi}{ds_\lambda} = -M \sin i \sin \alpha\lambda, \tag{7}$$

in which, bearing in mind that we have $\xi \times \lambda = 0$ along the whole ray Γ, indeed it results that ξ satisfies the equation

$$\frac{d\xi}{ds_\lambda} = M(\lambda \wedge \zeta) \wedge \xi = (\lambda \wedge \mathbf{p}) \wedge \xi, \tag{8}$$

that is Eq. (3*).

5. We observe that Eq. (1) can be interpreted as the relation between the ordinary covariant derivatives $\left(\frac{\delta\xi}{\delta s_\lambda}\right)$ and the derivatives in the sense of Weyl $\left(\frac{\delta\xi}{\delta s_\lambda}\right)$ relative to the *metric connection* that has $2p_i$ as components [7]. Hence, the result stated in the precedent number[, *in section 4.*,] can also be expressed in this way:

The luminous vector of a linearly polarized ray Γ, which propagates through a medium of variable refraction index $n(x, y, z)$, is transported along Γ through the parallelism relative to the metric connection (in the sense of Weyl) in $\mathbb{R}_3$[, connection] that has for its components the (covariant) components of the vector $\text{grad} \log n^2$ [8].

6. We can also consider the more general case in which the medium N is equipped with a *rotary power* for the polarized light [9]. If such rotary power is a function of $\tau(x, y, z)$ un point of N, we immediately get

$$\frac{d\xi}{ds_\lambda} = -M \sin i \sin \alpha\lambda + \tau\lambda \wedge \xi, \tag{9}$$

Hence, the luminous vector ξ is transported along Γ through parallelism with respect to C, meaning that to this variety one should attribute the metric determination defined by $dt = nds$ and that, moreover, also a *torsion* (in the meaning given by Cartan), function of the point of $\mathbb{C}_3$ and equal to 2τ [10].

Is it possible to assign a torsion (i.e., a rotary power) such that along Γ the vectors $\nu = \text{vers}(\lambda \wedge \zeta)$ (binormal to Γ) result being parallel in $\mathbb{C}_3$? And thus *the rays polarized in the in the incidence plane will be conserved as such*, which generally does not happen[*?*] This case, which in general seems not to be discarded *a priori*, cannot occur for τ *function of the point* P in $\mathbb{R}_3$. Indeed it is possible to see immediately that a condition for which this can occur is that the rotary power be $\tau = \frac{1}{T}$, *torsion* of Γ in $\mathbb{R}_3$ in a generic point. The invariant $\frac{1}{T}$ can be easily calculated: one finds that

$$\frac{1}{T} = -\frac{1}{M \sin^2 i} \epsilon^{irs} \lambda_i p_r p_{s/t} \lambda^t; \tag{10}$$

now this is a function of P and also of λ; and the corresponding transport of the ξ vectors *is not linear.*

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[1] At the meeting of the 5th of December, 1926.

[2] Schouten, *Der Ricci Kalkul*, Berlin, Springer, 1924, p. 168.

[3] Schouten, *ibid.*; see also Levi Civita, *Lectures of Absolute Differential Calculus*, Rome, Stock, 1925, p. 237 [*The Absolute Differential Calculus (calculus of Tensors)*].

[4] See for example Battelli and Cardani, *Treatise of Experimental Physics*, vol. II, Vallardi, 1913, p. 681.

[5] See fir example Levi-Civita, *Questions de Mecànica clàssica i relativistica*, Barcelona, 1921; Conf. IV, p. 126;

Duhem, *Sur le principe d'Optique géométrique énouncé per Fermat*, «Journal de Mathém.», 6th ser., vol. 8, 1912, pp. 1-58.

[6] As Duhem observed (*loc. cit.*), Fermat's principle is not always true; yet it is always true the following property, relative to the rays' trajectories.

[7] Weyl, *Raum, Zeit, Materie*, French ed., Paris, Blanchard, 1922, p. 108

[8] It is not necessary saying that such metric connection (in the sense of Weyl) results with a *segmentary curvature* (=*Streckenkrümmung*, according to Weyl[*'s definition*]; *curvature of a hometic*[*transformation*], according to Cartan[*'s definition*]) is everywhere null.

[9] See for example *loc. cit.* Eq. (4), p. 814. Extending the usual definition (relative to a homogenous and isotropic medium) we could say *rotary power* in P in the direction of (dP), the fraction (a derivative) $\frac{d\varphi}{ds}$, where $d\varphi$ is the angle of which the projection of the luminous vector gets rotated on the plane normal in P to the ray, when ξ is transported from P to $P + dP$.

[10] Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*, «Compte Rendus de l'Ac.», vol. 174, 1922, pp. 437-439; *Sur les variétés à connexion affine et la Théorie de la relativité generalisée*, «Annales de l'Ec. Norm. Sup. Rendus de l'Ac.», 3rd ser,, vol. 40, 1923, pp. 325-412, vol. 41, 1924, pp. 1-25, vol. 42, 1925, pp. 17-88; *Les recented généralisation de la notion d'espace* («Bull. des Sciences Math.», vol. 48, 1924, pp. 294-320), spec. pp. 303-304.